Numerical Analysis (10th Edition)

Chapter 8.2, Problem 10E

Problem

Repeat Exercise 3 using the results of Exercise 7.

Reference: Exercise 3

Find the least squares polynomial approximation of degree two to the functions and intervals in Exercise 1.

Reference: Exercise 1

Find the linear least squares polynomial approximation to *f (x)* on the indicated interval if

a. $f(x) = x^2 + 3x + 2, \quad [0, 1];$ **b.** $f(x) = x^3, \quad [0, 2$

c. $f(x) = \frac{1}{x}, \quad [1, 3];$ **d.** $f(x) = e^x, \quad [0, 2$

e. $f(x) = \frac{1}{2}\cos x + \frac{1}{3}\sin 2x, \quad [0, 1];$ **f.** $f(x) = x \ln x, \quad [$

Reference: Exercise 7

Use the Gram-Schmidt process to construct ∅0(x), ∅1(x), ∅2(x), and ∅3(x) for the following intervals.

a. [0, 1] **b.** [0, 2] **c.** [1, 3]

Step-by-step solution

Step 1 of 30

(a)

Consider the function $f(x)=x^2+3x+2$ and the interval $[0,1]$.

We know that in the Gram-Schmidt process the set of polynomials $\{\phi_0,\phi_1,\phi_2,\ldots,\phi_n\}$ defined as

$$\phi_0(x)=1$$
$$\phi_1(x)=x-B_1$$
$$where$$
$$B_1=\frac{\int_a^b xw(x)[\phi_0(x)]^2\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$

And when $k\ge 2$,

$$\phi_k(x)=(x-B_k)\phi_{k-1}(x)-C_k\phi_{k-2}(x)$$
$$where$$
$$B_k=\frac{\int_a^b xw(x)[\phi_{k-1}(x)]^2\,dx}{\int_a^b w(x)[\phi_{k-1}(x)]^2\,dx}$$
$$and$$
$$C_k=\frac{\int_a^b xw(x)\phi_{k-1}(x)\phi_{k-2}(x)\,dx}{\int_a^b w(x)[\phi_{k-2}(x)]^2\,dx}\ for x\in[a,b]$$

Comment

Step 2 of 30

The given interval is $[0,1]$

Let us take $w(x)=1$

$$\phi_0(x)=1$$
$$\phi_1(x)=x-B_1$$
$$where$$
$$B_1=\frac{\int_0^1 xw(x)[\phi_0(x)]^2\,dx}{\int_0^1 w(x)[\phi_0(x)]^2\,dx}$$
$$\Rightarrow B_1=\frac{\int_0^1 xdx}{\int_0^1 dx}$$
$$\Rightarrow B_1=\frac{\left[\frac{x^2}{2}\right]_0^1}{[x]_0^1}$$
$$\Rightarrow B_1=\frac{1}{2}$$

Thus

$$\phi_1(x)=x-B_1$$
$$\Rightarrow\phi_1(x)=x-\frac{1}{2}$$

Therefore,

$$\phi_1(x)=x-\frac{1}{2}$$
$$\phi_2(x)=(x-B_2)\phi_1(x)-C_2\phi_0(x)$$
$$where$$
$$B_2=\frac{\int_0^1 xw(x)[\phi_1(x)]^2\,dx}{\int_0^1 w(x)[\phi_1(x)]^2\,dx}$$
$$C_2=\frac{\int_0^1 xw(x)\phi_1(x)\phi_0(x)\,dx}{\int_0^1 w(x)[\phi_0(x)]^2\,dx}$$

Comment

Step 3 of 30

Now,

$$B_2=\frac{\int_0^1 x\left[x-\frac{1}{2}\right]^2dx}{\int_0^1\left[x-\frac{1}{2}\right]^2dx}$$
$$=\frac{\int_0^1 x\left[x^2+\frac{1}{4}-x\right]dx}{\int_0^1\left[x^2+\frac{1}{4}-x\right]dx}$$
$$=\frac{\int_0^1\left[x^3+\frac{x}{4}-x^2\right]dx}{\int_0^1\left[x^2+\frac{1}{4}-x\right]dx}$$
$$=\frac{\left[\frac{x^4}{4}+\frac{x^2}{8}-\frac{x^3}{3}\right]_0^1}{\left[\frac{x^3}{3}+\frac{x}{4}-\frac{x^2}{2}\right]_0^1}$$
$$=\frac{\left[\frac{x^4}{4}+\frac{x^2}{8}-\frac{x^3}{3}\right]_0^1}{\left[\frac{x^3}{3}+\frac{x}{4}-\frac{x^2}{2}\right]_0^1}$$
$$B_2=\frac{1}{2}$$

Comment

Step 4 of 30

And also,

$$C_2=\frac{\int_0^1 x\left[x-\frac{1}{2}\right]dx}{\int_0^1 dx}$$
$$=\frac{\int_0^1\left[x^2-\frac{x}{2}\right]dx}{\int_0^1 dx}$$
$$=\frac{\left[\frac{x^3}{3}-\frac{x^2}{4}\right]_0^1}{[x]_0^1}$$
$$C_2=\frac{1}{12}$$

Thus,

$$\phi_2(x)=(x-B_2)\phi_1(x)-C_2\phi_0(x)$$
$$\Rightarrow\phi_2(x)=\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2}\right)-\frac{1}{12}$$
$$\Rightarrow\phi_2(x)=\left(x-\frac{1}{2}\right)^2-\frac{1}{12}$$

We know that in the Gram-Schmidt process if $\{\phi_0,\phi_1,\phi_2,\ldots,\phi_n\}$ is an orthogonal set of functions on the interval $[a,b]$ with respect to the weight function w, then the least squares approximation to f on $[a,b]$ is defined as

$$P(x)=\sum_{j=0}^{n}a_j\phi_j(x),\ j=0,1,2,\ldots,n$$
$$where$$
$$a_j=\frac{\int_a^b w(x)\phi_j(x)f(x)\,dx}{\int_a^b w(x)[\phi_j(x)]^2\,dx}$$

Now the linear least squares polynomial approximation to $f(x)=x^2+3x+2$ on $[0,1]$ is defined as

$$P(x)=\sum_{j=0}^{1}a_j\phi_j(x),\ j=0,1$$
$$\Rightarrow P(x)=a_0\phi_0(x)+a_1\phi_1(x)$$
$$where$$
$$a_0=\frac{\int_a^b w(x)\phi_0(x)f(x)\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$
$$a_1=\frac{\int_a^b w(x)\phi_1(x)f(x)\,dx}{\int_a^b w(x)[\phi_1(x)]^2\,dx}$$
$$a_2=\frac{\int_a^b w(x)\phi_2(x)f(x)\,dx}{\int_a^b w(x)[\phi_2(x)]^2\,dx}$$

Comment

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Now,

$$a_0=\frac{\int_a^b w(x)\phi_0(x)f(x)\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$
$$\Rightarrow a_0=\frac{\int_0^1(x^2+3x+2)\,dx}{\int_0^1 dx}$$
$$\Rightarrow a_0=\frac{\left[\frac{x^3}{3}+\frac{3x^2}{2}+2x\right]_0^1}{[x]_0^1}$$
$$\Rightarrow a_0=3.8333$$
$$a_1=\frac{\int_a^b w(x)\phi_1(x)f(x)\,dx}{\int_a^b w(x)[\phi_1(x)]^2\,dx}$$
$$\Rightarrow a_1=\frac{\int_0^1\phi_1(x)f(x)\,dx}{\int_0^1[\phi_1(x)]^2\,dx}$$
$$\Rightarrow a_1=\frac{\int_0^1\left(x-\frac{1}{2}\right)(x^2+3x+2)\,dx}{\int_0^1\left[x-\frac{1}{2}\right]^2dx}$$
$$\Rightarrow a_1=\frac{\int_0^1\left(x^3+\frac{5x^2}{2}+\frac{x}{2}-1\right)dx}{\int_0^1\left[x-\frac{1}{2}\right]^2dx}$$
$$\Rightarrow a_1=\frac{3\left(\frac{x^4}{4}+\frac{5x^3}{6}+\frac{x^2}{4}-x\right)_0^1}{\left[\left(x-\frac{1}{2}\right)^3\right]_0^1}$$
$$\Rightarrow a_1=4$$

Comment

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And

$$a_2=\frac{\int_a^b w(x)\phi_2(x)f(x)\,dx}{\int_a^b w(x)[\phi_2(x)]^2\,dx}$$
$$a_2=\frac{\int_0^1\left(\left(x-\frac{1}{2}\right)^2-\frac{1}{12}\right)(x^2+3x+2)\,dx}{\int_0^1\left[\left(x-\frac{1}{2}\right)^2-\frac{1}{12}\right]^2dx}$$
$$a_2=\frac{\int_0^1\left(x^4+2x^3-\frac{5x^2}{6}-\frac{3x}{2}+\frac{1}{3}\right)dx}{\int_0^1\left[x^4+\frac{2x^2}{3}-2x^3+\frac{x}{3}+\frac{1}{36}\right]dx}$$
$$a_2=\frac{\left(\frac{x^5}{5}+\frac{x^4}{2}-\frac{5x^3}{18}-\frac{3x^2}{4}+\frac{x}{3}\right)_0^1}{\left[\frac{x^5}{5}+\frac{2x^3}{9}-\frac{x^4}{2}+\frac{x^2}{6}+\frac{x}{36}\right]_0^1}$$
$$a_2=\frac{5.5555\times10^{-3}}{0.1167}$$
$$a_2=0.0476$$

Thus the linear least squares polynomial approximation of degree two to the given function is:

$$P(x)=a_0\phi_0(x)+a_1\phi_1(x)+a_2\phi_2(x)$$
$$\Rightarrow P(x)=3.8333+4\left(x-\frac{1}{2}\right)+0.0476\left[\left(x-\frac{1}{2}\right)^2-\frac{1}{12}\right]$$

Therefore,

$$\boxed{P(x)=3.8333+4\left(x-\frac{1}{2}\right)+0.0476\left[\left(x-\frac{1}{2}\right)^2-\frac{1}{12}\right]}$$

Comment

Step 7 of 30

(b)

Consider the function $f(x)=x^3$ and the interval $[0,2]$.

We know that in the Gram-Schmidt process the set of polynomials $\{\phi_0,\phi_1,\phi_2,\ldots,\phi_n\}$ defined as

Comment

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$$\phi_0(x)=1$$
$$\phi_1(x)=x-B_1$$
$$where$$
$$B_1=\frac{\int_a^b xw(x)[\phi_0(x)]^2\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$

And when $k\ge 2$,

$$\phi_k(x)=(x-B_k)\phi_{k-1}(x)-C_k\phi_{k-2}(x)$$
$$where$$
$$B_k=\frac{\int_a^b xw(x)[\phi_{k-1}(x)]^2\,dx}{\int_a^b w(x)[\phi_{k-1}(x)]^2\,dx}$$
$$and$$
$$C_k=\frac{\int_a^b xw(x)\phi_{k-1}(x)\phi_{k-2}(x)\,dx}{\int_a^b w(x)[\phi_{k-2}(x)]^2\,dx}\ for x\in[a,b]$$

Comment

Step 9 of 30

The given interval is $[0,2]$

Let us take $w(x)=1$

$$\phi_0(x)=1$$
$$\phi_1(x)=x-B_1$$
$$where$$
$$B_1=\frac{\int_0^2 xw(x)[\phi_0(x)]^2\,dx}{\int_0^2 w(x)[\phi_0(x)]^2\,dx}$$
$$\Rightarrow B_1=\frac{\int_0^2 xdx}{\int_0^2 dx}$$
$$\Rightarrow B_1=\frac{\left[\frac{x^2}{2}\right]_0^2}{[x]_0^2}$$
$$\Rightarrow B_1=1$$

Thus

$$\phi_1(x)=x-B_1$$
$$\Rightarrow\phi_1(x)=x-1$$

Therefore

$$\phi_1(x)=x-1$$
$$\phi_2(x)=(x-B_2)\phi_1(x)-C_2\phi_0(x)$$
$$where$$
$$B_2=\frac{\int_0^2 xw(x)[\phi_1(x)]^2\,dx}{\int_0^2 w(x)[\phi_1(x)]^2\,dx}$$
$$C_2=\frac{\int_0^2 xw(x)\phi_1(x)\phi_0(x)\,dx}{\int_0^2 w(x)[\phi_0(x)]^2\,dx}$$

Comment

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Now

$$B_2=\frac{\int_0^2 x[x-1]^2\,dx}{\int_0^2[x-1]^2\,dx}$$
$$\Rightarrow B_2=\frac{\int_0^1 x\left[x^2+1-2x\right]dx}{\int_0^1\left[x^2+1-2x\right]dx}$$
$$\Rightarrow B_2=\frac{\int_0^1\left[x^3+x-2x^2\right]dx}{\int_0^1\left[x^2+1-2x\right]dx}$$
$$\Rightarrow B_2=\frac{\left[\frac{x^4}{4}+\frac{x^2}{2}-\frac{2x^3}{3}\right]_0^2}{\left[\frac{x^3}{3}+x-x^2\right]_0^2}$$
$$\Rightarrow B_2=\frac{\left[\frac{x^4}{4}+\frac{x^2}{8}-\frac{x^3}{3}\right]_0^2}{\left[\frac{x^3}{3}+x-x^2\right]_0^2}$$
$$\Rightarrow B_2=\frac{\left(2/3\right)}{\left(2/3\right)}$$
$$\Rightarrow B_2=1$$

$$\Rightarrow C_2 = \frac{\int_0^2 \left[x^2 - x\right] dx}{\int_0^2 dx}$$

$$\Rightarrow C_2 = \frac{\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^2}{[x]_0^2}$$

$$\Rightarrow C_2 = \frac{\left(\frac{2}{3}\right)}{2}$$

$$\Rightarrow C_2 = \frac{1}{3}$$

Thus,

$$\phi_2(x) = (x - B_2)\phi_1(x) - C_2\phi_0(x)$$
$$\Rightarrow \phi_2(x) = (x-1)(x-1) - \frac{1}{3}$$
$$\Rightarrow \phi_2(x) = (x-1)^2 - \frac{1}{3}$$

Comment

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We know that in the Gram-Schmidt process if $\{\phi_0, \phi_1, \phi_2, ..., \phi_n\}$ is an orthogonal set of functions on the interval $[a,b]$ with respect to the weight function w, then the least squares approximation to f on $[a,b]$ is defined as

$$P(x) = \sum_{j=0}^{n} a_j\phi_j(x), j = 0,1,2,...,n$$

where

$$a_j = \frac{\int_a^b w(x)\phi_j(x) f(x) dx}{\int_a^b w(x)\left[\phi_j(x)\right]^2 dx}$$

Now the linear least squares polynomial approximation to $f(x) = x^3$ on $[0,2]$ is defined as

$$P(x) = \sum_{j=0}^{1} a_j\phi_j(x), j = 0,1$$
$$\Rightarrow P(x) = a_0\phi_0(x) + a_1\phi_1(x)$$

where

$$a_0 = \frac{\int_a^b w(x)\phi_0(x) f(x) dx}{\int_a^b w(x)\left[\phi_0(x)\right]^2 dx}$$

$$a_1 = \frac{\int_a^b w(x)\phi_1(x) f(x) dx}{\int_a^b w(x)\left[\phi_1(x)\right]^2 dx}$$

$$a_2 = \frac{\int_a^b w(x)\phi_2(x) f(x) dx}{\int_a^b w(x)\left[\phi_2(x)\right]^2 dx}$$

Now,

$$a_0 = \frac{\int_a^b w(x)\phi_0(x) f(x) dx}{\int_a^b w(x)\left[\phi_0(x)\right]^2 dx}$$

$$\Rightarrow a_0 = \frac{\int_0^2 (x^3) dx}{\int_0^2 dx}$$

$$\Rightarrow a_0 = \frac{\left[\frac{x^4}{4}\right]_0^2}{[x]_0^2}$$

$$\Rightarrow a_0 = 2$$

$$a_1 = \frac{\int_a^b w(x)\phi_1(x) f(x) dx}{\int_a^b w(x)\left[\phi_1(x)\right]^2 dx}$$

$$\Rightarrow a_1 = \frac{\int_0^2 \phi_1(x) f(x) dx}{\int_0^2 \left[\phi_1(x)\right]^2 dx}$$

$$\Rightarrow a_1 = \frac{\int_0^2 (x-1)(x^3) dx}{\int_0^2 [x-1]^2 dx}$$

$$\Rightarrow a_1 = \frac{3\left[\left(\frac{x^5}{5} - \frac{x^4}{4}\right)\right]_0^2}{\left([x-1]^3\right)_0^2}$$

$$\Rightarrow a_1 = \frac{18}{5}$$
$$\Rightarrow a_1 = 3.6$$

Comment

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And also,

$$a_2 = \frac{\int_a^b w(x)\phi_2(x) f(x) dx}{\int_a^b w(x)\left[\phi_2(x)\right]^2 dx}$$

$$a_2 = \frac{\int_0^2 (x^3)\left[(x-1)^2 - \frac{1}{3}\right] dx}{\int_0^2 \left[(x-1)^2 - \frac{1}{3}\right]^2 dx}$$

$$a_2 = \frac{\int_0^2 (x^3)\left[x^2 + \frac{2}{3} - 2x\right] dx}{\int_0^2 \left[x^2 + \frac{2}{3} - x\right]^2 dx}$$

$$a_2 = \frac{\int_0^2 \left[x^5 + \frac{2}{3}x^3 - 2x^4\right] dx}{\int_0^2 \left[x^4 - \frac{20}{9} + \frac{16x^2}{3} - 4x^3\right] dx}$$

$$a_2 = \frac{\left[\frac{x^6}{6} + \frac{2x^4}{12} - \frac{2x^5}{5}\right]_0^2}{\left[\frac{x^5}{5} - \frac{20x}{9} + \frac{16x^3}{9} - x^4\right]_0^2}$$

$$a_2 = \frac{[0.5333]}{[0.1777]}$$

$$a_2 = 3$$

Thus the linear least squares polynomial approximation to the given function is:

$$P(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x)$$
$$\Rightarrow P(x) = 2 + 3.6(x-1) + 3\left[(x-1)^2 - \frac{1}{3}\right]$$

Therefore,

$$\boxed{P(x) = 2 + 3.6(x-1) + 3\left[(x-1)^2 - \frac{1}{3}\right]}$$

Comment

Step 13 of 30

(c)

Consider the function $f(x) = \frac{1}{x}$ and the interval $[1,3]$.

We know that in the Gram-Schmidt process the set of polynomials $\{\phi_0, \phi_1, \phi_2, ..., \phi_n\}$ defined as

Comment

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$$\phi_0(x) = 1$$
$$\phi_1(x) = x - B_1$$

where

$$B_1 = \frac{\int_a^b xw(x)\left[\phi_0(x)\right]^2 dx}{\int_a^b w(x)\left[\phi_0(x)\right]^2 dx}$$

And when $k \ge 2,$

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k\phi_{k-2}(x)$$

where

$$B_k = \frac{\int_a^b xw(x)\left[\phi_{k-1}(x)\right]^2 dx}{\int_a^b w(x)\left[\phi_{k-1}(x)\right]^2 dx}$$

and

$$C_k = \frac{\int_a^b xw(x)\phi_{k-1}(x)\phi_{k-2}(x) dx}{\int_a^b w(x)\left[\phi_{k-2}(x)\right]^2 dx} \; for x \in [a,b]$$

The given interval is $[1,3]$

Let us take $w(x) = 1$

$$\phi_0(x) = 1$$
$$\phi_1(x) = x - B_1$$

where

$$B_1 = \frac{\int_1^3 xw(x)\left[\phi_0(x)\right]^2 dx}{\int_1^3 w(x)\left[\phi_0(x)\right]^2 dx}$$

$$\Rightarrow B_1 = \frac{\int_1^3 x dx}{\int_1^3 dx}$$

$$\Rightarrow B_1 = \frac{\left[\frac{x^2}{2}\right]_1^3}{[x]_1^3}$$

$$\Rightarrow B_1 = 2$$

Thus

$$\phi_1(x) = x - B_1$$
$$\Rightarrow \phi_1(x) = x - 2$$

Therefore

$$\phi_1(x) = x - 2$$
$$\phi_2(x) = (x - B_2)\phi_1(x) - C_2\phi_0(x)$$

where

$$B_2 = \frac{\int_1^3 xw(x)\left[\phi_1(x)\right]^2 dx}{\int_1^3 w(x)\left[\phi_1(x)\right]^2 dx}$$

$$C_2 = \frac{\int_1^3 xw(x)\phi_1(x)\phi_0(x) dx}{\int_1^3 w(x)\left[\phi_0(x)\right]^2 dx}$$

Comment

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Now,

$$B_2 = \frac{\int_1^3 x[x-2]^2 dx}{\int_1^3 [x-2]^2 dx}$$

$$\Rightarrow B_2 = \frac{\int_1^3 x\left[x^2 + 4 - 4x\right] dx}{\int_1^3 \left[x^2 + 4 - 4x\right] dx}$$

$$\Rightarrow B_2 = \frac{\int_1^3 \left[x^3 + 4x - 4x^2\right] dx}{\int_1^3 \left[x^2 + 4 - 4x\right] dx}$$

$$\Rightarrow B_2 = \frac{\left[\frac{x^4}{4} + 2x^2 - \frac{4x^3}{3}\right]_1^3}{\left[\frac{x^3}{3} + 4x - 2x^2\right]_1^3}$$

$$\Rightarrow B_2 = \frac{1.333}{0.666}$$
$$\Rightarrow B_2 = 2$$

$$C_2 = \frac{\int_1^3 x[x-2] dx}{\int_1^3 dx}$$

$$\Rightarrow C_2 = \frac{\int_1^3 \left[x^2 - 2x\right] dx}{\int_1^3 dx}$$

$$\Rightarrow C_2 = \frac{\left[\frac{x^3}{3} - x^2\right]_1^3}{[x]_1^3}$$

$$\Rightarrow C_2 = \frac{1}{3}$$

Thus

$$\phi_2(x) = (x - B_2)\phi_1(x) - C_2\phi_0(x)$$
$$\Rightarrow \phi_2(x) = (x-2)(x-2) - \frac{1}{3}$$
$$\Rightarrow \phi_2(x) = (x-2)^2 - \frac{1}{3}$$

We know that in the Gram-Schmidt process if $\{\phi_0, \phi_1, \phi_2, ..., \phi_n\}$ is an orthogonal set of functions on the interval $[a,b]$ with respect to the weight function w, then the least squares approximation to f on $[a,b]$ is defined as

$$P(x) = \sum_{j=0}^{n} a_j\phi_j(x), j = 0,1,2,...,n$$

where

$$a_j = \frac{\int_a^b w(x)\phi_j(x) f(x) dx}{\int_a^b w(x)\left[\phi_j(x)\right]^2 dx}$$

Now the linear least squares polynomial approximation to $f(x) = \frac{1}{x}$ on $[1,3]$ is defined as

$$P(x) = \sum_{j=0}^{1} a_j\phi_j(x), j = 0,1$$
$$\Rightarrow P(x) = a_0\phi_0(x) + a_1\phi_1(x)$$

where

$$a_0 = \frac{\int_a^b w(x)\phi_0(x) f(x) dx}{\int_a^b w(x)\left[\phi_0(x)\right]^2 dx}$$

$$a_1 = \frac{\int_a^b w(x)\phi_1(x) f(x) dx}{\int_a^b w(x)\left[\phi_1(x)\right]^2 dx}$$

$$a_2 = \frac{\int_a^b w(x)\phi_2(x) f(x) dx}{\int_a^b w(x)\left[\phi_2(x)\right]^2 dx}$$

Comment

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Now

$$a_0 = \frac{\int_a^b w(x)\phi_0(x) f(x) dx}{\int_a^b w(x)\left[\phi_0(x)\right]^2 dx}$$

$$\Rightarrow a_0 = \frac{\int_1^3 \left(\frac{1}{x}\right) dx}{\int_1^3 dx}$$

$$\Rightarrow a_0 = \frac{[\log 3 - \log 1]}{2}$$

$$\Rightarrow a_0 = 0.23856$$

$$a_1 = \frac{\int_a^b w(x)\phi_1(x) f(x) dx}{\int_a^b w(x)\left[\phi_1(x)\right]^2 dx}$$

$$\Rightarrow a_1 = \frac{\int_0^2 \phi_1(x) f(x) dx}{\int_0^2 \left[\phi_1(x)\right]^2 dx}$$

$$\Rightarrow a_1 = \frac{\int_1^3 (x-2)\left(\frac{1}{x}\right) dx}{\int_1^3 [x-2]^2 dx}$$

$$\Rightarrow a_1 = 3 \times \frac{\int_1^3 \left(1 - \frac{2}{x}\right) dx}{\left([x-2]^3\right)_1^3}$$

$$\Rightarrow a_1 = 3 \times \frac{\left([x - 2\log x]\right)_1^3}{\left([x-2]^3\right)_1^3}$$

$$\Rightarrow a_1 = 1.56863$$

$$a_2 = \frac{\int_a^b w(x)\phi_2(x) f(x) dx}{\int_a^b w(x)\left[\phi_2(x)\right]^2 dx}$$

$$\Rightarrow a_2 = \frac{\int_1^3 \left[(x-2)^2 - \frac{1}{3}\right]\left(\frac{1}{x}\right) dx}{\int_1^3 \left[(x-2)^2 - \frac{1}{3}\right]^2 dx}$$

$$\Rightarrow a_2 = \frac{\int_1^3 \left[x - 4 + \frac{11}{3x}\right] dx}{\int_1^3 \left[x^2 + \frac{11}{3} - 4x\right]^2 dx}$$

$$\Rightarrow a_2 = \frac{\int_1^3 \left[x - 4 + \frac{11}{3x}\right] dx}{\int_1^3 \left[x^4 + \frac{70x^2}{3} - 8x^3 - \frac{88x}{3} + 13.444\right] dx}$$

$$\Rightarrow a_2 = \frac{\left[\frac{x^2}{2} - 4x + \frac{11}{3}\log x\right]_1^3}{\left[\frac{x^5}{5} + \frac{70x^3}{9} - \frac{8x^4}{4} - \frac{44x^2}{3} + 13.444x\right]_1^3}$$

$$\Rightarrow a_2 = \frac{[0.02825]}{0.17688}$$

$$\Rightarrow a_2 = 0.15971$$

Thus the linear least squares polynomial approximation to the given function is:

$$P(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x)$$
$$\Rightarrow P(x) = 0.23856 + 1.56863(x-2) + 0.15971\left[(x-2)^2 - \frac{1}{3}\right]$$

Therefore

$$\boxed{P(x) = 0.23856 + 1.56863(x-2) + 0.15971\left[(x-2)^2 - \frac{1}{3}\right]}$$

Comment

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(d)

Consider the function $f(x) = e^x$ and the interval $[0,2]$.

We know that in the Gram-Schmidt process the set of polynomials $\{\phi_0, \phi_1, \phi_2, ..., \phi_n\}$ defined as

$$\phi_0(x) = 1$$
$$\phi_1(x) = x - B_1$$

where

$$B_1 = \frac{\int_a^b xw(x)\left[\phi_0(x)\right]^2 dx}{\int_a^b w(x)\left[\phi_0(x)\right]^2 dx}$$

And when $k \ge 2,$

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k\phi_{k-2}(x)$$

where

$$B_k = \frac{\int_a^b xw(x)\left[\phi_{k-1}(x)\right]^2 dx}{\int_a^b w(x)\left[\phi_{k-1}(x)\right]^2 dx}$$

and

$$C_k = \frac{\int_a^b xw(x)\phi_{k-1}(x)\phi_{k-2}(x) dx}{\int_a^b w(x)\left[\phi_{k-2}(x)\right]^2 dx} \; for x \in [a,b]$$

The given interval is $[0,2]$

Let us take $w(x) = 1$

Comment

Step 18 of 30

$$\phi_0(x) = 1$$
$$\phi_1(x) = x - B_1$$

where

$$B_1 = \frac{\int_0^2 xw(x)\left[\phi_0(x)\right]^2 dx}{\int_0^2 w(x)\left[\phi_0(x)\right]^2 dx}$$

$$\int_0^2 dx$$

$$\Rightarrow B_1=\frac{\left[\frac{x^2}{2}\right]_0^2}{[x]_0^2}$$

$$\Rightarrow B_1=1$$

Thus

$$\phi_1(x)=x-B_1$$
$$\Rightarrow \phi_1(x)=x-1$$

Therefore

$$\phi_1(x)=x-1$$
$$\phi_2(x)=(x-B_2)\phi_1(x)-C_2\phi_0(x)$$
$$where$$
$$B_2=\frac{\int_0^2 xw(x)[\phi_1(x)]^2\,dx}{\int_0^2 w(x)[\phi_1(x)]^2\,dx}$$
$$C_2=\frac{\int_0^2 xw(x)\phi_1(x)\phi_0(x)\,dx}{\int_0^2 w(x)[\phi_0(x)]^2\,dx}$$

Comment

Step 19 of 30

Now,

$$B_2=\frac{\int_0^2 x[x-1]^2\,dx}{\int_0^2 [x-1]^2\,dx}$$

$$\Rightarrow B_2=\frac{\int_0^1 x\left[x^2+1-2x\right]dx}{\int_0^1\left[x^2+1-2x\right]dx}$$

$$\Rightarrow B_2=\frac{\int_0^1\left[x^3+x-2x^2\right]dx}{\int_0^1\left[x^2+1-2x\right]dx}$$

$$\Rightarrow B_2=\frac{\left[\frac{x^4}{4}+\frac{x^2}{2}-\frac{2x^3}{3}\right]_0^2}{\left[\frac{x^3}{3}+x-x^2\right]_0^2}$$

$$\Rightarrow B_2=\frac{\left[\frac{x^4}{4}+\frac{x^2}{8}-\frac{x^3}{3}\right]_0^2}{\left[\frac{x^3}{3}+x-x^2\right]_0^2}$$

$$\Rightarrow B_2=\frac{\left(\frac{2}{3}\right)}{\left(\frac{2}{3}\right)}$$

$$\Rightarrow B_2=1$$

$$C_2=\frac{\int_0^2 x[x-1]\,dx}{\int_0^2 dx}$$

$$\Rightarrow C_2=\frac{\int_0^2\left[x^2-x\right]dx}{\int_0^2 dx}$$

$$\Rightarrow C_2=\frac{\left[\frac{x^3}{3}-\frac{x^2}{2}\right]_0^2}{[x]_0^2}$$

$$\Rightarrow C_2=\frac{\left(\frac{2}{3}\right)}{2}$$

$$\Rightarrow C_2=\frac{1}{3}$$

Comment

Step 20 of 30

Thus

$$\phi_2(x)=(x-B_2)\phi_1(x)-C_2\phi_0(x)$$
$$\Rightarrow \phi_2(x)=(x-1)(x-1)-\frac{1}{3}$$
$$\Rightarrow \phi_2(x)=(x-1)^2-\frac{1}{3}$$

We know that in the Gram-Schmidt process if $\{\phi_0,\phi_1,\phi_2,\ldots,\phi_n\}$ is an orthogonal set of functions on the interval $[a,b]$ with respect to the weight function w, then the least squares approximation to f on $[a,b]$ is defined as

$$P(x)=\sum_{j=0}^{n}a_j\phi_j(x),\ j=0,1,2,\ldots,n$$
$$where$$
$$a_j=\frac{\int_a^b w(x)\phi_j(x)f(x)\,dx}{\int_a^b w(x)[\phi_j(x)]^2\,dx}$$

Now the linear least squares polynomial approximation to $f(x)=e^x$ on $[0,2]$ is defined as

$$P(x)=\sum_{j=0}^{1}a_j\phi_j(x),\ j=0,1$$
$$\Rightarrow P(x)=a_0\phi_0(x)+a_1\phi_1(x)$$
$$where$$
$$a_0=\frac{\int_a^b w(x)\phi_0(x)f(x)\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$
$$a_1=\frac{\int_a^b w(x)\phi_1(x)f(x)\,dx}{\int_a^b w(x)[\phi_1(x)]^2\,dx}$$
$$a_2=\frac{\int_a^b w(x)\phi_2(x)f(x)\,dx}{\int_a^b w(x)[\phi_2(x)]^2\,dx}$$

Comment

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Now

$$a_0=\frac{\int_a^b w(x)\phi_0(x)f(x)\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$

$$\Rightarrow a_0=\frac{\int_0^2 (e^x)\,dx}{\int_0^2 dx}$$

$$\Rightarrow a_0=\frac{\left[e^x\right]_0^2}{[x]_0^2}$$

$$\Rightarrow a_0=\frac{e^2-1}{2}$$

$$\Rightarrow a_0=3.1945$$

$$a_1=\frac{\int_a^b w(x)\phi_1(x)f(x)\,dx}{\int_a^b w(x)[\phi_1(x)]^2\,dx}$$

$$\Rightarrow a_1=\frac{\int_0^2 \phi_1(x)f(x)\,dx}{\int_0^2[\phi_1(x)]^2\,dx}$$

$$\Rightarrow a_1=\frac{\int_0^2 (x-1)(e^x)\,dx}{\int_0^2[x-1]^2\,dx}$$

$$\Rightarrow a_1=3\times\frac{\left[e^x(x-1)\right]_0^2-\int_0^2(e^x)\,dx}{\left([x-1]^3\right)_0^2}$$

$$\Rightarrow a_1=3$$

$$a_2=\frac{\int_a^b w(x)\phi_2(x)f(x)\,dx}{\int_a^b w(x)[\phi_2(x)]^2\,dx}$$

$$\Rightarrow a_2=\frac{\int_0^2(e^x)\left((x-1)^2-\frac{1}{3}\right)dx}{\int_0^2\left[(x-1)^2-\frac{1}{3}\right]^2dx}$$

$$\Rightarrow a_2=\frac{\left[(e^x)\left((x-1)^2-\frac{1}{3}\right)\right]_0^2-2\int_0^2(e^x)(x-1)\,dx}{\int_0^2\left[x^2+\frac{2}{3}-2x\right]^2dx}$$

$$\Rightarrow a_2=\frac{\left[(e^2-1)\left(\frac{2}{3}\right)\right]-2\int_0^2(e^x)(x-1)\,dx}{\int_0^2\left[x^4-\frac{20}{9}+\frac{16x^2}{3}-4x^3\right]dx}$$

$$\Rightarrow a_2=\frac{\left[(e^2-1)\left(\frac{2}{3}\right)\right]-4}{\left[\frac{x^5}{5}-\frac{20x}{9}+\frac{16x^3}{9}-x^4\right]_0^2}$$

$$\Rightarrow a_2=\frac{0.25937}{0.1777}$$

$$\Rightarrow a_2=1.45902$$

Thus the linear least squares polynomial approximation to the given function is:

$$P(x)=a_0\phi_0(x)+a_1\phi_1(x)+a_2\phi_2(x)$$
$$\Rightarrow P(x)=3.1945+3(x-1)+1.45902\left[(x-1)^2-\frac{1}{3}\right]$$

Therefore

$$\boxed{P(x)=3.1945+3(x-1)+1.45902\left[(x-1)^2-\frac{1}{3}\right]}$$

Comment

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(e)

Consider the function $f(x)=\frac{1}{2}\cos x+\frac{1}{3}\sin 2x$ and the interval $[0,1]$.

We know that in the Gram-Schmidt process the set of polynomials $\{\phi_0,\phi_1,\phi_2,\ldots,\phi_n\}$ defined as

$$\phi_0(x)=1$$
$$\phi_1(x)=x-B_1$$
$$where$$
$$B_1=\frac{\int_a^b xw(x)[\phi_0(x)]^2\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$

And when $k\ge 2$,

$$\phi_k(x)=(x-B_k)\phi_{k-1}(x)-C_k\phi_{k-2}(x)$$
$$where$$
$$B_k=\frac{\int_a^b xw(x)[\phi_{k-1}(x)]^2\,dx}{\int_a^b w(x)[\phi_{k-1}(x)]^2\,dx}$$
$$and$$
$$C_k=\frac{\int_a^b xw(x)\phi_{k-1}(x)\phi_{k-2}(x)\,dx}{\int_a^b w(x)[\phi_{k-2}(x)]^2\,dx}\ for x\in[a,b]$$

The given interval is $[0,1]$

Let us take $w(x)=1$

$$\phi_0(x)=1$$
$$\phi_1(x)=x-B_1$$
$$where$$
$$B_1=\frac{\int_0^1 xw(x)[\phi_0(x)]^2\,dx}{\int_0^1 w(x)[\phi_0(x)]^2\,dx}$$

$$\Rightarrow B_1=\frac{\int_0^1 x\,dx}{\int_0^1 dx}$$

$$\Rightarrow B_1=\frac{\left[\frac{x^2}{2}\right]_0^1}{[x]_0^1}$$

$$\Rightarrow B_1=\frac{1}{2}$$

Comment

Step 23 of 30

Thus

$$\phi_1(x)=x-B_1$$
$$\Rightarrow \phi_1(x)=x-\frac{1}{2}$$

Therefore

$$\phi_1(x)=x-\frac{1}{2}$$
$$\phi_2(x)=(x-B_2)\phi_1(x)-C_2\phi_0(x)$$
$$where$$
$$B_2=\frac{\int_0^1 xw(x)[\phi_1(x)]^2\,dx}{\int_0^1 w(x)[\phi_1(x)]^2\,dx}$$
$$C_2=\frac{\int_0^1 xw(x)\phi_1(x)\phi_0(x)\,dx}{\int_0^1 w(x)[\phi_0(x)]^2\,dx}$$

Now

$$B_2=\frac{\int_0^1 x\left[x-\frac{1}{2}\right]^2dx}{\int_0^1\left[x-\frac{1}{2}\right]^2dx}$$

$$\Rightarrow B_2=\frac{\int_0^1 x\left[x^2+\frac{1}{4}-x\right]dx}{\int_0^1\left[x^2+\frac{1}{4}-x\right]dx}$$

$$\Rightarrow B_2=\frac{\int_0^1\left[x^3+\frac{x}{4}-x^2\right]dx}{\int_0^1\left[x^2+\frac{1}{4}-x\right]dx}$$

$$\Rightarrow B_2=\frac{\left[\frac{x^4}{4}+\frac{x^2}{8}-\frac{x^3}{3}\right]_0^1}{\left[\frac{x^3}{3}+\frac{x}{4}-\frac{x^2}{2}\right]_0^1}$$

$$\Rightarrow B_2=\frac{\left[\frac{x^4}{4}+\frac{x^2}{8}-\frac{x^3}{3}\right]_0^1}{\left[\frac{x^3}{3}+\frac{x}{4}-\frac{x^2}{2}\right]_0^1}$$

$$\Rightarrow B_2=\frac{1}{2}$$

$$C_2=\frac{\int_0^1 x\left[x-\frac{1}{2}\right]dx}{\int_0^1 dx}$$

$$\Rightarrow C_2=\frac{\int_0^1\left[x^2-\frac{x}{2}\right]dx}{\int_0^1 dx}$$

$$\Rightarrow C_2=\frac{\left[\frac{x^3}{3}-\frac{x^2}{4}\right]_0^1}{[x]_0^1}$$

$$\Rightarrow C_2=\frac{1}{12}$$

Thus,

$$\phi_2(x)=(x-B_2)\phi_1(x)-C_2\phi_0(x)$$
$$\Rightarrow \phi_2(x)=\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2}\right)-\frac{1}{12}$$
$$\Rightarrow \phi_2(x)=\left(x-\frac{1}{2}\right)^2-\frac{1}{12}$$

We know that in the Gram-Schmidt process if $\{\phi_0,\phi_1,\phi_2,\ldots,\phi_n\}$ is an orthogonal set of functions on the interval $[a,b]$ with respect to the weight function w, then the least squares approximation to f on $[a,b]$ is defined as

$$P(x)=\sum_{j=0}^{n}a_j\phi_j(x),\ j=0,1,2,\ldots,n$$
$$where$$
$$a_j=\frac{\int_a^b w(x)\phi_j(x)f(x)\,dx}{\int_a^b w(x)[\phi_j(x)]^2\,dx}$$

Comment

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Now the linear least squares polynomial approximation to $f(x)=\frac{1}{2}\cos x+\frac{1}{3}\sin 2x$ on $[0,1]$ is defined as

$$P(x)=\sum_{j=0}^{1}a_j\phi_j(x),\ j=0,1$$
$$\Rightarrow P(x)=a_0\phi_0(x)+a_1\phi_1(x)$$
$$where$$
$$a_0=\frac{\int_a^b w(x)\phi_0(x)f(x)\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$
$$a_1=\frac{\int_a^b w(x)\phi_1(x)f(x)\,dx}{\int_a^b w(x)[\phi_1(x)]^2\,dx}$$
$$a_2=\frac{\int_a^b w(x)\phi_2(x)f(x)\,dx}{\int_a^b w(x)[\phi_2(x)]^2\,dx}$$

Now

$$a_0=\frac{\int_a^b w(x)\phi_0(x)f(x)\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$

$$\Rightarrow a_0=\frac{\int_0^1\left(\frac{1}{2}\cos x+\frac{1}{3}\sin 2x\right)dx}{\int_0^1 dx}$$

$$\Rightarrow a_0=\frac{\left[\frac{1}{2}\sin x-\frac{1}{6}\cos 2x\right]_0^1}{[x]_0^1}$$

$$\Rightarrow a_0=0.65676$$

$$a_1=\frac{\int_a^b w(x)\phi_1(x)f(x)\,dx}{\int_a^b w(x)[\phi_1(x)]^2\,dx}$$

$$\Rightarrow a_1=\frac{\int_0^1\phi_1(x)f(x)\,dx}{\int_0^1[\phi_1(x)]^2\,dx}$$

$$\Rightarrow a_1=\frac{\int_0^1\left(x-\frac{1}{2}\right)\left(\frac{1}{2}\cos x+\frac{1}{3}\sin 2x\right)dx}{\int_0^1\left[x-\frac{1}{2}\right]^2dx}$$

$$\Rightarrow a_1=\frac{\left[\left(x-\frac{1}{2}\right)\left(\frac{1}{2}\sin x-\frac{1}{6}\cos 2x\right)\right]_0^1-\int_0^1\left(\left(\frac{1}{2}\sin x-\frac{1}{6}\cos 2x\right)\right)dx}{\int_0^1\left[x-\frac{1}{2}\right]^2dx}$$

$$\Rightarrow a_1=\frac{\left[\left(\frac{1}{2}\times 0.49\right)-\left(\frac{1}{12}\right)\right]-\left(-\frac{1}{2}\cos x-\frac{1}{12}\sin 2x\right)_0^1}{\int_0^1\left[x-\frac{1}{2}\right]^2dx}$$

$$\Rightarrow a_1=12\times(0.16166-0.15407)$$

$$\Rightarrow a_1=0.09103$$

$$a_2=\frac{\int_a^b w(x)\phi_2(x)f(x)\,dx}{\int_a^b w(x)[\phi_2(x)]^2\,dx}$$

$$a_2=\frac{\int_0^1\left(\left(x-\frac{1}{2}\right)^2-\frac{1}{12}\right)\left(\frac{1}{2}\cos x+\frac{1}{3}\sin 2x\right)dx}{\int_0^1\left[\left(x-\frac{1}{2}\right)^2-\frac{1}{12}\right]^2dx}$$

$$a_2=\frac{\left[\left(\left(x-\frac{1}{2}\right)^2-\frac{1}{12}\right)\left(\frac{1}{2}\sin x-\frac{1}{6}\cos 2x\right)\right]_0^1-2\int_0^1\left(\left(x-\frac{1}{2}\right)\left(\frac{1}{2}\sin x-\frac{1}{6}\cos 2x\right)\right)dx}{\int_0^1\left[x^2-\frac{1}{6}-x\right]^2dx}$$

$$a_2=\frac{[(0.08168-0.02777)]-2\int_0^1\left(\left(x-\frac{1}{2}\right)\left(\frac{1}{2}\sin x-\frac{1}{6}\cos 2x\right)\right)dx}{\int_0^1\left[x^4+\frac{2x^2}{3}-2x^3+\frac{x}{3}+\frac{1}{36}\right]dx}$$

$$a_2=\frac{[(0.05391)]-2\left[\left[\left(x-\frac{1}{2}\right)\left(-\frac{1}{2}\cos x-\frac{1}{12}\sin 2x\right)\right]_0^1-\int_0^1\left(-\frac{1}{2}\cos x-\frac{1}{12}\sin 2x\right)dx\right]}{\left[\frac{x^5}{5}+\frac{2x^3}{9}-\frac{x^4}{2}+\frac{x^2}{6}+\frac{x}{36}\right]_0^1}$$

$$a_2 = \frac{[(0.05391)] + [[(0.84592)] - 2(0.43807 + 0.041667)]}{0.11667}$$
$$a_2 = \frac{-0.05964}{0.11667}$$
$$a_2 = -0.51122$$

Thus the linear least squares polynomial approximation to the given function is:

$$P(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x)$$
$$\Rightarrow P(x) = 0.65676 + 0.09103\left(x - \frac{1}{2}\right) - 0.51122\left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{12}\right]$$

Therefore

$$\boxed{P(x) = 0.65676 + 0.09103\left(x - \frac{1}{2}\right) - 0.51122\left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{12}\right]}$$

Comment

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(f)

Consider the function $f(x) = x\ln x$ and the interval $[1,3]$.

We know that in the Gram-Schmidt process the set of polynomials $\{\phi_0, \phi_1, \phi_2, ..., \phi_n\}$ defined as

$$\phi_0(x) = 1$$
$$\phi_1(x) = x - B_1$$
$$\textit{where}$$
$$B_1 = \frac{\int_a^b xw(x)[\phi_0(x)]^2\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$

And when $k \geq 2$,

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k\phi_{k-2}(x)$$
$$\textit{where}$$
$$B_k = \frac{\int_a^b xw(x)[\phi_{k-1}(x)]^2\,dx}{\int_a^b w(x)[\phi_{k-1}(x)]^2\,dx}$$
$$\textit{and}$$
$$C_k = \frac{\int_a^b xw(x)\phi_{k-1}(x)\phi_{k-2}(x)\,dx}{\int_a^b w(x)[\phi_{k-2}(x)]^2\,dx} \textit{ for } x \in [a,b]$$

Comment

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The given interval is $[1,3]$

Let us take $w(x) = 1$

$$\phi_0(x) = 1$$
$$\phi_1(x) = x - B_1$$
$$\textit{where}$$
$$B_1 = \frac{\int_1^3 xw(x)[\phi_0(x)]^2\,dx}{\int_1^3 w(x)[\phi_0(x)]^2\,dx}$$
$$\Rightarrow B_1 = \frac{\int_1^3 x\,dx}{\int_1^3 dx}$$
$$\Rightarrow B_1 = \frac{\left[\frac{x^2}{2}\right]_1^3}{[x]_1^3}$$
$$\Rightarrow B_1 = 2$$

Thus

$$\phi_1(x) = x - B_1$$
$$\Rightarrow \phi_1(x) = x - 2$$

Comment

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Therefore,

$$\phi_1(x) = x - 2$$
$$\phi_2(x) = (x - B_2)\phi_1(x) - C_2\phi_0(x)$$
$$\textit{where}$$
$$B_2 = \frac{\int_1^3 xw(x)[\phi_1(x)]^2\,dx}{\int_1^3 w(x)[\phi_1(x)]^2\,dx}$$
$$C_2 = \frac{\int_1^3 xw(x)\phi_1(x)\phi_0(x)\,dx}{\int_1^3 w(x)[\phi_0(x)]^2\,dx}$$

Now

$$B_2 = \frac{\int_1^3 x[x-2]^2\,dx}{\int_1^3 [x-2]^2\,dx}$$
$$\Rightarrow B_2 = \frac{\int_1^3 x[x^2 + 4 - 4x]\,dx}{\int_1^3 [x^2 + 4 - 4x]\,dx}$$
$$\Rightarrow B_2 = \frac{\int_1^3 [x^3 + 4x - 4x^2]\,dx}{\int_1^3 [x^2 + 4 - 4x]\,dx}$$
$$\Rightarrow B_2 = \frac{\left[\frac{x^4}{4} + 2x^2 - \frac{4x^3}{3}\right]_1^3}{\left[\frac{x^3}{3} + 4x - 2x^2\right]_1^3}$$
$$\Rightarrow B_2 = \frac{1.333}{0.666}$$
$$\Rightarrow B_2 = 2$$
$$C_2 = \frac{\int_1^3 x[x-2]\,dx}{\int_1^3 dx}$$
$$\Rightarrow C_2 = \frac{\int_1^3 [x^2 - 2x]\,dx}{\int_1^3 dx}$$
$$\Rightarrow C_2 = \frac{\left[\frac{x^3}{3} - x^2\right]_1^3}{[x]_1^3}$$
$$\Rightarrow C_2 = \frac{1}{3}$$

Comment

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Thus

$$\phi_2(x) = (x - B_2)\phi_1(x) - C_2\phi_0(x)$$
$$\Rightarrow \phi_2(x) = (x-2)(x-2) - \frac{1}{3}$$
$$\Rightarrow \phi_2(x) = (x-2)^2 - \frac{1}{3}$$

We know that in the Gram-Schmidt process if $\{\phi_0, \phi_1, \phi_2, ..., \phi_n\}$ is an orthogonal set of functions on the interval $[a,b]$ with respect to the weight function w, then the least squares approximation to f on $[a,b]$ is defined as

$$P(x) = \sum_{j=0}^{n} a_j\phi_j(x), j = 0,1,2,...,n$$
$$\textit{where}$$
$$a_j = \frac{\int_a^b w(x)\phi_j(x)f(x)\,dx}{\int_a^b w(x)[\phi_j(x)]^2\,dx}$$

Now the linear least squares polynomial approximation to $f(x) = x\ln x$ on $[1,3]$ is defined as

$$P(x) = \sum_{j=0}^{1} a_j\phi_j(x), j = 0,1$$
$$\Rightarrow P(x) = a_0\phi_0(x) + a_1\phi_1(x)$$
$$\textit{where}$$
$$a_0 = \frac{\int_a^b w(x)\phi_0(x)f(x)\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$
$$a_1 = \frac{\int_a^b w(x)\phi_1(x)f(x)\,dx}{\int_a^b w(x)[\phi_1(x)]^2\,dx}$$
$$a_2 = \frac{\int_a^b w(x)\phi_2(x)f(x)\,dx}{\int_a^b w(x)[\phi_2(x)]^2\,dx}$$

Now

$$a_0 = \frac{\int_a^b w(x)\phi_0(x)f(x)\,dx}{\int_a^b w(x)[\phi_0(x)]^2\,dx}$$
$$\Rightarrow a_0 = \frac{\int_1^3 (x\ln x)\,dx}{\int_1^3 dx}$$
$$\Rightarrow a_0 = \frac{\left[\ln x \times \frac{x^2}{2}\right]_1^3 - \frac{1}{2}\int_1^3 (x)\,dx}{2}$$
$$\Rightarrow a_0 = 1.47188$$

Comment

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And also,

$$a_1 = \frac{\int_a^b w(x)\phi_1(x)f(x)\,dx}{\int_a^b w(x)[\phi_1(x)]^2\,dx}$$
$$\Rightarrow a_1 = \frac{\int_0^2 \phi_1(x)f(x)\,dx}{\int_0^2 [\phi_1(x)]^2\,dx}$$
$$\Rightarrow a_1 = \frac{\int_1^3 (x-2)(x\ln x)\,dx}{\int_1^3 [x-2]^2\,dx}$$
$$\Rightarrow a_1 = 3 \times \frac{\int_1^3 (x^2 - 2x)\ln x\,dx}{\left([x-2]^3\right)_1^3}$$
$$\Rightarrow a_1 = 3 \times \frac{\left(\left[\left(\frac{x^3}{3} - x^2\right)\ln x\right]\right)_1^3 - \int_1^3 \left(\frac{x^2}{3} - x\right)dx}{\left([x-2]^3\right)_1^3}$$
$$\Rightarrow a_1 = -\frac{3}{2} \times \left(\left[\left(\frac{x^3}{9} - \frac{x^2}{2}\right)\right]\right)_1^3$$
$$\Rightarrow a_1 = \frac{5}{3}$$
$$\Rightarrow a_1 = 1.6667$$

Comment

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Now,

$$a_2 = \frac{\int_a^b w(x)\phi_2(x)f(x)\,dx}{\int_a^b w(x)[\phi_2(x)]^2\,dx}$$
$$a_2 = \frac{\int_1^3 \left[(x-2)^2 - \frac{1}{3}\right](x\ln x)\,dx}{\int_1^3 \left[(x-2)^2 - \frac{1}{3}\right]^2 dx}$$
$$a_2 = \frac{\int_1^3 \left[x^3 + \frac{11}{3}x - 4x^2\right](\ln x)\,dx}{\int_1^3 \left[x^2 + \frac{11}{3} - 4x\right]^2 dx}$$
$$a_2 = \frac{\left(\left[\frac{x^4}{4} + \frac{11x^2}{6} - 4\frac{x^3}{3}\right](\ln x)\right)_1^3 - \int_1^3 \left[\frac{x^3}{4} + \frac{11x}{6} - 4\frac{x^2}{3}\right]dx}{\int_1^3 \left[x^4 + \frac{70x^2}{3} - 8x^3 - \frac{88x}{3} + 13.444\right]dx}$$
$$a_2 = \frac{(0.82396) - \left[\frac{x^4}{16} + \frac{11x^2}{12} - 4\frac{x^3}{9}\right]_1^3}{\left[\frac{x^5}{5} + \frac{70x^3}{9} - \frac{8x^4}{4} - \frac{44x^2}{3} + 13.444x\right]_1^3}$$
$$a_2 = \frac{(0.82396) - [1.3125 - 0.53472]}{0.17688}$$
$$a_2 = \frac{0.04618}{0.17688}$$
$$a_2 = 0.26108$$

Thus the linear least squares polynomial approximation to the given function is:

$$P(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x)$$
$$\Rightarrow P(x) = 1.47188 + 1.6667(x-2) + 0.26108\left[(x-2)^2 - \frac{1}{3}\right]$$

Therefore,

$$\boxed{P(x) = 1.47188 + 1.6667(x-2) + 0.26108\left[(x-2)^2 - \frac{1}{3}\right]}$$

Comment